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Equivalent Conditions for n-equivalences Modulo a Class c of Groups

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Abstract. In this work we define the notion of a map $f : X \to Y$ to be an n-equivalence modulo a class $\mathcal{C}$ of groups. Then we show an equivalent condition, which is more close to a homological condition, in order to a map $f : X \to Y$ to be an n-equivalence modulo a class $\mathcal{C}$ of groups. Finally, at least for a complex K which is finite and is a suspension of a connected space, the notion above is also given in terms of the map $f_{\#} : [K, X] \to [K, Y]$.

0. Introduction

In [W2], J.H.C. Whitehead gives the definition of a map $f : X \to Y$ to be an n-equivalence. Then he shows that $f : X \to Y$ is an n-equivalence if and only if $\pi_i(X) \xrightarrow{f_{i\#}} \pi_i(Y)$ is a isomorphism for $i \leq n$. Motivated by above we define that $f : X \to Y$ is an n-equivalence mod a class $\mathcal{C}$ of groups if $\pi_i(X) \xrightarrow{f_{i\#}} \pi_i(Y)$ is a $\mathcal{C}$-isomorphisms for $i \leq n$ whenever this makes sense. I remark that one can show that if X, Y are H-spaces and f is an H-map or K is a suspension of a connected space then $[K, X] \to [K, Y]$ is a $\mathcal{C}$-isomorphism if $dim(K) \leq n$ where K is a complex and $\mathcal{C}$ is a class of Nilpotent groups. The precise formulation and the proof of this result is given in the appendix. This result also has motivated us to define $f : X \to Y$ to be an n-equivalence mod a class $\mathcal{C}$ as above.

In part I and II we will show that the condition $\pi_i(X) \to \pi_i(Y)$ is a $\mathcal{C}$-isomorphism for $i \leq n$ can be given almost in terms of homological conditions at least if we assume that X, Y satisfy certain hypotheses. In fact our result is given in terms of a generalized class of groups. More

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precisely we prove: Theorem 2.1. Let X, Y be $\mathcal{C}$-nilpotent spaces. If $\mathcal{C}$ is an acyclic class and either a) X, Y are of finite type and $H_*(\pi, A) \in \mathcal{C}$, $* > 0$ for $\pi, A \in \mathcal{C}$ or b) $\mathcal{C}$ is complete, then the following two conditions below are equivalent

a) $\pi_i(X) \to \pi_i(Y)$ is a $\mathcal{C}$-isomorphism for $i \leq n$,

b) $H_i(X) \to H_i(Y)$ is a $\mathcal{C}$-isomorphism for $i \leq n$ and $\psi_{n+1} : H_{n+1}(X) \oplus \pi_{n+1}(Y) \to H_{n+1}(Y)$ is a $\mathcal{C}$-epimorphism. Here $\psi_{n+1} = (f_{n+1}, h_{n+1})$ where h_{n+1} is the Hurewicz map.

We would like to point out that a) implies b) if we assume that $\pi_1(X) \to \pi_1(Y)$ is surjective and $\mathcal{C}$ is acyclic and complete without further hypotheses on the spaces X and Y. This is theorem 1.1. Otherwise the result is not true according with examples at the end of part I. Finally we would like to point out that under the hypotheses of theorem 2.1, in the appendix we show that the projection $\pi_{n+1}(Y, X) \to \pi'_{n+1}(Y, X)$ is a $\mathcal{C}$-isomorphism if we assume that $\pi_1(X) \in \mathcal{C}$, where $\pi'_{n+1}(Y, X)$ is the quotient of $\pi_{n+1}(Y, X)$ by the action of $\pi_1(X)$. We believe this fact his its own interest, although it is not necessary in the proof of theorem 2.1. For the definition and details about $\mathcal{C}$-nilpotent spaces and generalized classes of groups, see [G].

1. Part I: One side of the relative Hurewicz Theorem mod $\mathcal{C}$

Given a pair (Y, X) of connected spaces, it is well known that if $\pi_i(Y, X) = 0$, $1 \leq i \leq n$ then $H_i(Y, X) = 0$, $1 \leq i \leq n$ and $h'_{n+1} : \pi'_{n+1}(Y, X) \to H_{n+1}(Y, X)$ is an isomorphism. One would like to known if this result is true in terms of class. By a class $\mathcal{C}$ of groups we mean an abelian class of groups or a generalized class of groups. See [S_1] resp [G] for the definitions and details about class of groups. The answer is yes but we must have some hypothesis on the class of groups $\mathcal{C}$ and the spaces in question. Namely:

Theorem 1.1. *Let $\mathcal{C}$ be a class of groups which is acyclic and complete then if $\pi_1(X) \to \pi_1(Y)$ is surjective and $\pi_i(Y, X) \in \mathcal{C}$, $1 \leq i \leq n$, we have $H_i(Y, X) \in \mathcal{C}$, $1 \leq i \leq n$ and $h'_{n+1} : \pi'_{n+1}(Y, X) \to H_{n+1}(Y, X)$ is a $\mathcal{C}$-isomorphism.*

In order to prove this result we will need three lemmas, where we assume the hypotheses of the theorem.

Lemma 1.2. *Let π be a group which acts on A. If $A \in \mathcal{C}$ then $H_*(\pi, A) \in \mathcal{C}$, $* \geq 0$.*

Proof. This is similar to Proposition 1.4 of [G].

Lemma 1.3. *Let π acts on the abelian groups A and B. If $\psi : A \to B$ is a $\mathcal{C}$-isomorphism of π-modules, then $H_*(\pi, A) \to H_*(\pi, B)$ is a $\mathcal{C}$-isomorphism, $* \geq 0$.*

Proof. Let us consider the two short exact sequences:

$$0 \to ker\,\psi \to A \to \psi(A) \to 0 \qquad (I)$$

$$0 \to \psi(A) \to B \to \frac{B}{\psi(A)} \to 0 \qquad (II)$$

By the long exact sequences in homology associated with the sequence of coefficients (I) and (II) together with Lemma 1.2 we get

$$H_*(\pi, A) \approx H_*(\pi, \psi(A))$$

and

$$H_*(\pi, \psi(A)) \approx H_*(\pi, \frac{B}{\psi(A)}).$$

Then the result follows. □

Lemma 1.4. *Let Y be any connected space and $A \in \mathcal{C}$. Then $H_*(Y, A) \in \mathcal{C}$, $* \geq 0$.*

Proof. The proof can be done using the Postnikov system of Y plus the previous Lemma 1.2. Another proof can be given as follows: given Y, we know by [K,T] Theorem 1.1 that we can associate a group denoted by π_Y such that $H_*(Y, A) \approx H_*(\pi_Y, A)$, where A is any local coefficient system and the last homology means group homology. Then by Lemma 1.2 above the result follows. □

Proof of Theorem 1.1. Let us assume that $\pi_1(X) \to \pi_1(Y)$ is an isomorphism. Then we consider the fibration $(\tilde{Y}, \tilde{X}) \to (Y, X) \to K(\pi, 1)$, where $\pi = \pi_1(X)$, and $\tilde{Y}$ respectively $\tilde{X}$ are the universal cover of Y respectively X. Since $\pi_i(\tilde{Y}, \tilde{X}) \in \mathcal{C}$, $2 \leq i \leq n$, by the Relative Hurewicz

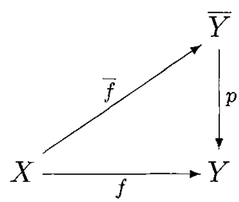
Theorem modulo a Serre class, see [S1], we have $H_i(\tilde{Y},\tilde{X}) \in \mathcal{C}$, $2 \le i \le n$ and $h_{n+1} : \pi_{n+1}(\tilde{Y},\tilde{X}) \to H_{n+1}(\tilde{Y},\tilde{X})$ is a $\mathcal{C}$-isomorphism. By Serre's spectral sequence associated with the above fibration we have:

$$H_p(\pi, H_q(\tilde{Y},\tilde{X})) \Longrightarrow H_{p+q}(Y,X)$$

Since $H_p(\pi, H_q(\tilde{Y},\tilde{X})) \in \mathcal{C}$ for $q \le n$ by Lemma 1.2, we have that $H_i(Y,X) \in \mathcal{C}$ for $1 \le i \le n$ and $H_0(\pi, H_{n+1}(\tilde{Y},\tilde{X})) \longrightarrow H_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism.

By the argument given in the last part of the proof of Proposition 3.6 of [G], which is in the page 60, we can replace $H_{n+1}(\tilde{Y},\tilde{X})$ by $\pi_{n+1}(Y,X)$ and it follows that $h'_{n+1} : \pi'_{n+1}(Y,X) \to H_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism.

If $\pi_1(X) \to \pi_1(Y)$ is not injective we consider the next stage of the Moore-Postnikov decomposition of the map $f : X \to Y$, see [S2]. Then we have the diagram:



The fibre of p is $K(\pi_2(Y,X);1)$. Since the class $\mathcal{C}$ is acyclic

$$\tilde{H}_*(K(\pi_2(Y,X),1);\mathbb{Z}) \in \mathcal{C}, \quad * > 0,$$

and by Lemma 1.4 $H_p\left(Y, H_q(K(\pi_2(Y,X),1);\mathbb{Z})\right) \in \mathcal{C}$ for $q > 0$. Then it follows that $p_* : H_*(\overline{Y},\mathbb{Z}) \to H_*(Y,\mathbb{Z})$ is a $\mathcal{C}$-isomorphism for $* \ge 0$. Since $p_\# : \pi_r(\overline{Y}) \to \pi_r(Y)$ is a $\mathcal{C}$-isomorphism for all r, it suffices to show the result for $\overline{f}$. By the first part the result follows. □

Remark. The hypotheses that $\mathcal{C}$ be complete and $\pi_1(X) \to \pi_1(Y)$ be surjective are necessary. For let us consider two examples.

Example 1. Suppose that $\pi_1(X) \to \pi_1(Y)$ is only $\mathcal{C}$-surjective where $\mathcal{C}$ is the class of groups of odd torsion. Let $X = \tilde{Y}$ and Y a space such that

$$\pi_i(Y) = \begin{cases} \mathbb{Z}_3 & i = 1 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i = n \\ O & \text{otherwise} \end{cases}$$

and the action of $\mathbb{Z}_3$ on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Such a space Y exists by [S$_2$] exercise A-3 p.460. By the spectral sequence associated with the universal cover of Y we have

$$H_n(Y, \mathbb{Z}) \approx H_0(\pi, H_n(\tilde{Y}, \mathbb{Z}) = H_0(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \approx 0.$$

By the long exact sequence of the pair $(Y, \tilde{Y})$ we have

$$H_{n+1}(Y, \mathbb{Z}) \to H_{n+1}(Y, \tilde{Y}; \mathbb{Z}) \to H_n(\tilde{Y}, \mathbb{Z}) \to H_n(Y, \mathbb{Z})$$

or

$$H_{n+1}(Y, \tilde{Y}, \mathbb{Z}) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0.$$

So $\pi_{n+1}(Y, \tilde{Y}) \in \mathcal{C}$ and $H_{n+1}(Y, \tilde{Y}, \mathbb{Z}) \notin \mathcal{C}$. So h'_{n+1} can't be a $\mathcal{C}$-isomorphism.

Example 2. Let $X = \prod_{i=1}^{\infty} S_i^2$, $Y = S^2 \times X$ then $\pi_i(Y, X) \in \mathcal{C}$ where $\mathcal{C}$ is the class of the finitely generated groups. But $H_2(Y, X, \mathbb{Z}) \notin \mathcal{C}$. This type of example has been pointed out in [S$_1$].

2. Part II: The n-th homotopy type of a space and the Hurewicz map

The purpose of this section is to show:

Theorem 2.1. *Let $\mathcal{C}$ be an acyclic class, (Y, X) a pair where X and Y are connected $\mathcal{C}$-nilpotent spaces. If a) X, Y are of finite type and $H_*(\pi, A) \in \mathcal{C}$, $* > 0$ for $\pi, A \in \mathcal{C}$, or b) $\mathcal{C}$ is complete, then the two conditions below are equivalent for $n \geq 1$.*

a$_n$) *$\pi_l(X) \to \pi_l(Y)$ is a $\mathcal{C}$-isomorphism for $l \leq n$,* b$_n$) *$H_l(X) \to H_l(Y)$ is a $\mathcal{C}$-isomorphism for $l \leq n$ and $((i_{n+1})_*, h_{n+1})$: $H_{n+1}(X) \oplus \pi_{n+1}(Y) \to H_{n+1}(Y)$ is $\mathcal{C}$-surjective, where i is the inclusion map from X to Y.*

Remark 1. If we do not assume Y and X to be $\mathcal{C}$-nilpotent spaces it is still true that a$_n$) implies b$_n$) under the conditions that the hypotheses b) holds and $\pi_1(X) \to \pi_1(Y)$ is surjective. The proof is the same as the proof of Theorem 2.1 where we use Theorem 1.1 of Part 1.

Remark 2. One of the main ingredients of the proof of Theorem 2.1 is the Relative Hurewicz Theorem. In fact under the hypotheses of Theorem 2.1 we show, in the appendix, that the natural projection $\pi_{n+1}(Y,X) \to \pi'_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism if we assume that $\pi_1(X) \in \mathcal{C}$. Although, we do not use this fact, in the proof of Theorem 2.1, may have its own interest.

Proof of Theorem 2.1. $(a_n) \Rightarrow b_n$. Let us consider the diagram where the rows are long exact sequence of the pair (Y,X) in homology and homotopy:

$$\begin{array}{ccccccccccc}
\to \pi_{n+1}(X) & \xrightarrow{(i_{n+1})_\#} & \pi_{n+1}(Y) & \xrightarrow{(j_{n+1})_\#} & \pi_{n+1}(Y,X) & \xrightarrow{\partial_{n+1}} & \pi_n(X) & \xrightarrow{(i_n)_\#} & \pi_n(Y) \to \\
\downarrow h_{n+1} & & \downarrow h_{n+1} & & \downarrow h_{n+1} & & \downarrow h_{n+1} & & \downarrow h_{n+1} \\
\to H_{n+1}(X) & \xrightarrow{(i_{n+1})_*} & H_{n+1}(Y) & \xrightarrow{(j_{n+1})_*} & H_{n+1}(Y,X) & \xrightarrow{\partial_{n+1}} & H_n(X) & \xrightarrow{(i_n)_*} & H_n(Y) \to
\end{array}$$

From the hypothesis a$_n$) we have $\pi_i(Y,X) \in \mathcal{C}$, $i < n+1$. By the Relative Hurewicz Theorem, see [G] p.55, we have $H_i(Y,X) \in \mathcal{C}$, $i < n+1$ and $h'_{n+1} : \pi'_{n+1}(Y,X) \to H_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism. Since $\pi_{n+1}(Y,X) \to \pi'_{n+1}(Y,X)$ is surjective we also have that $h_{n+1} : \pi_{n+1}(Y,X) \to H_{n+1}(Y,X)$ is $\mathcal{C}$-surjective. Since $im(h_{n+1}) \hookrightarrow H_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism we that $\partial_{n+1}(im\, h_{n+1})$ is $\mathcal{C}$-isomorphism to $\partial_{n+1}(H_{n+1}(Y,X))$. But $\partial_{n+1}(im\, h_{n+1}) = h_n(\partial_{n+1}(\pi_{n+1}(Y,X))$. Since $\partial_{n+1}(\pi_{n+1}(Y,X)) \in \mathcal{C}$ by the hypothesis a$_n$), follows that $(i_n)_*$ is $\mathcal{C}$-injective. Therefore $H_l(X) \to H_l(Y)$ is a $\mathcal{C}$-isomorphism for $1 \leq l \leq n$.

It remains to show that $(i_{n+1})_*, h_{n+1}$: $H_{n+1}(X) \oplus \pi_{n+1}(Y) \to H_{n+1}(Y)$ is $\mathcal{C}$-surjective. We have that $(j_{n+1})_\# : \pi_{n+1}(Y) \to \pi_{n+1}(Y,X)$

is $\mathcal{C}$-surjective because

$$\frac{\pi_{n+1}(Y,X)}{im(j_{n+1})_{\#}} \approx im(\partial_{n+1}) = ker(i_n)_{\#} \in \mathcal{C}$$

and $h_{n+1} \circ (j_{n+1})_{\#} = (j_{n+1})_* \circ h_{n+1}$. So $h_{n+1} \circ (j_{n+1})_{\#}$ and $(j_{n+1})_* \circ h_{n+1}$ are $\mathcal{C}$-surjective as well the map $(j_{n+1})_* \circ h_{n+1} : \pi_{n+1}(Y) \to im(j_{n+1})_*$. But

$$im(j_{n+1})_* \approx \frac{H_{n+1}(Y)}{ker(j_{n+1})_*} \approx \frac{H_{n+1}(Y)}{im(i_{n+1})_*}$$

and the result follows.

Now let's prove that $b_n) \Rightarrow a_n)$. From $b_n)$ we have that $H_l(Y,X) \in \mathcal{C}$, $l < n+1$. By the Relative Hurewicz Theorem, see [G] p.55 we have $\pi_l(Y,X) \in \mathcal{C}$, $l < n+1$. It remains to show that $\pi_n(X) \to \pi_n(Y)$ is $\mathcal{C}$-injective.

For this purpose, call $K = ker(i_n)_{\#} : \pi_n(X) \to Y_n(Y)$ and consider the exact sequence of π-modules where $\pi = \pi_1(X)$

$$\pi_{n+1}(Y) \xrightarrow{(j_{n+1})_{\#}} \pi_{n+1}(Y,X) \xrightarrow{\ p\ } K \longrightarrow 0$$

Now let us consider the sequence obtained from the one above modulo by the action of π:

$$\pi'_{n+1}(Y) \xrightarrow{(j'_{n+1})_{\#}} \pi'_{n+1}(Y,X) \xrightarrow{\ p'\ } K \longrightarrow 0$$

where M' denotes the quotient of M by the π-action, and the maps are the induced ones. This sequence is no longer exact but certainly p' is surjective and $p' \circ j'$ is the null homomorphism.

Also we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\to \pi_{n+1}(X) & \xrightarrow{(i_{n+1})_{\#}} & \pi_{n+1}(Y) & \xrightarrow{(j_{n+1})_{\#}} & \pi_{n+1}(Y,X) & \xrightarrow{\partial_{n+1}} & \pi_n(X) & \xrightarrow{(i_n)_{\#}} & \pi_n(Y) \to \\
& & \searrow & & \searrow & & & & \\
\downarrow h_{n+1} & & \downarrow h_{n+1} \quad \pi'_{n+1}(Y) & \longrightarrow & \pi'_{n+1}(Y,X) & \longrightarrow & K' \to 0 & & \downarrow h_n \\
& & \swarrow & & \downarrow h_{n+1} \quad \swarrow & & \downarrow h_n & & \\
\to H_{n+1}(X) & \xrightarrow{(i_{n+1})_*} & H_{n+1}(Y) & \xrightarrow{(j_{n+1})_*} & H_{n+1}(Y,X) & \xrightarrow{\partial_{n+1}} & H_n(X) & \xrightarrow{(i_n)_*} & H_n(Y) \to
\end{array}$$

where $\pi'_{n+1}(Y,X) \to H_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism. We are going to show that $K' \in \mathcal{C}$. For this it suffices to show that

$$\frac{\pi'_{n+1}(Y,X)}{im(j'_{n+1})_{\#}} \in \mathcal{C} \qquad \text{or} \qquad \frac{H_{n+1}(Y,X)}{im((j_{n+1})_* \circ h'_{n+1})} \in \mathcal{C}.$$

But

$$\begin{aligned} im((j_{n+1})_* \circ h'_{n+1}) &= im((j_{n+1})_* \circ h'_{n+1} + (j_{n+1})_* \circ (i_{n+1})_*) \\ &= im((j_{n+1})_* \circ (h'_{n+1} + (i_{n+1})_*)) \end{aligned}$$

which is $\mathcal{C}$-isomorphic to $im(j_{n+1})_*$ as result of the hypothesis that $((j_{n+1})_*, h_{n+1})$ is $\mathcal{C}$-surjective. So

$$\frac{H_{n+1}(Y,X)}{im((j_{n+1})_* \circ h'_{n+1})} \cong \frac{H_{n+1}(Y,X)}{im(j_{n+1})_*} \cong ker(H_n(X) \to H_n(Y)) \in \mathcal{C} .$$

Therefore $K' \in \mathcal{C}$. By proposition 3.3 of [G] we have that $K \in \mathcal{C}$ and the result follows. □

Appendix

Let $[K,X]$ denote the homotopy classes of base pointed maps. In general, $[K,X]$ does not have a group structure. So it does not make sense to say that a map $\varphi : [K,X] \to [K,Y]$ is $\mathcal{C}$-bijective. Nevertheless if X is a connected homotopy associative H-space, or K is a suspension of a connected space, by [W_1] chapter X , the set $[K,X]$ has a group structure. Further $[K,Y]$, in both cases, is a nilpotent group if K is a complex of finite dimension. Now we will prove a proposition which somehow justifies the definition of a map $f : X \to Y$ be a n-equivalence mod $\mathcal{C}$. For our purpose we can assume that $\mathcal{C}$ is a class of groups as described in $[H,R]$ or just replace $\mathcal{C}$ by the intersection of $\mathcal{C}$ with the category of all nilpotent groups. Let us assume that $\pi_1(X)$ and $\pi_1(Y)$ are $\mathcal{C}$-nilpotent groups. Of course in one of the cases where X, Y are H-spaces this hypothesis is automatically satisfied.

Proposition 1. *Let X, Y be connected homotopy associative H-spaces and $f : X \to Y$ be an H-map or K be a suspension of a connected space. If $f : X \to Y$ is an n-equivalence mod $\mathcal{C}$ then for any complex K where $dim\, K \leq n$ the induced map $\varphi_f : [K,X] \to [K,Y]$ is a $\mathcal{C}$-bijection if either*

a) *$\mathcal{C}$ is complete or*

b) *K is finite and $\mathcal{C}$ is proper.*

Proof. Let $n = 1$. Than K has the homotopy type of a bouquet of circles and $[K, X] = \bigoplus \pi_1(X)$ where the sum runs over a set J. Since $\pi_1(X) \to \pi_1(Y)$ is a $\mathcal{C}$-isomorphism the result follows either if a) or b) holds. Let us assume the result is true for maps which are $(n-1)$-equivalences mod $\mathcal{C}$. Take K where $dim\, K = n$. We have $\bigvee_J S^{n-1} \to K^{n-1} \to K$ where the first map comes from the construction of the CW complex K from the $n-1$-skeleton of K. The Puppe sequence associated with the above sequence and the induced homomorphisms induced by f, provide the commutative diagram below, where the rows are exact:

$$\begin{array}{ccccccccc}
[\Sigma K^{n-1}, X] & \longrightarrow & [\bigvee S^n, X] & \longrightarrow & [K, X] & \longrightarrow & [K^{n-1}, X] & \longrightarrow & [\bigvee S^{n-1}, X] \\
\downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 \\
[\Sigma K^{n-1}, Y] & \longrightarrow & [\bigvee S^n, Y] & \longrightarrow & [K, Y] & \longrightarrow & [K^{n-1}, Y] & \longrightarrow & [\bigvee S^{n-1}, Y]
\end{array}$$

The maps φ_1 and φ_4 are $\mathcal{C}$-isomorphisms by hypothesis. The map φ_0 is a $\mathcal{C}$-isomorphism by induction hypothesis applied to the map $\Omega f : \Omega X \to \Omega Y$ and the complex K^{n-1} of dimension $n-1$ since $[\sum K^{n-1}, X] \approx [K^{n-1}, \Omega X]$ and $[\sum K^{n-1}, Y] \approx [K^{n-1}, \Omega Y]$. Finally φ_3 is a $\mathcal{C}$-isomorphism by induction hypothesis. Now by the "$\mathcal{C}$-five lemma" Theorem 2.2 of $[H, R]$ applied to the class $\mathcal{C} \cap Nil$ where Nil is the category of the nilpotent groups, the result follows. □

For K a finite complex which is a suspension of a connected space, we know that $[K, Y]$ is a nilpotent group. Therefore, it makes sense to ask if $[K, X] \xrightarrow{f_\#} [K, Y]$ is a $\mathcal{C}$-bijection. For K a finite complex of $dim\, K \leq n$ and $f_\#$ a $\mathcal{C}$-bijection, then certainly $f : X \to Y$ is an n-equivalence mod $\mathcal{C}$, where we assume that $\pi_1(X), \pi_1(Y)$ are $\mathcal{C}$-nilpotent.

Now let us show that if $\pi_1(X) \in \mathcal{C}$ then in fact the Hurewicz homomorphism induces an isomorphism from $\pi_{n+1}(Y, X)$ into $H_{n+1}(Y, X)$ instead of $\pi'_{n+1}(Y, X)$.

This fact is not necessary for the proof of Theorem 2.2, but it is a natural generalization of the simply connected case and might be useful.

Proposition 2.1. *Let $\mathcal{C}$ be an acyclic class, (Y,X) a pair where X and Y are connected $\mathcal{C}$-nilpotent spaces and $\pi_1(X) \in \mathcal{C}$. If a) X, Y are of finite type and $H_*(\pi, A) \in \mathcal{C} > 0$ for $\pi, A \in \mathcal{C}$ or b) $\mathcal{C}$ is complete, then $\pi_{n+1}(Y,X) \to \pi'_{n+1}(Y,X)$ is a $\mathcal{C}$-isomorphism.*

The proof follows from the lemmas:

Lemma 3. *Let $\pi \in \mathcal{C}$ which acts $\mathcal{C}$-nilpotently on A where $\mathcal{C}$ is an acyclic class. If a) A is a finitely generated abelian group or b) $\mathcal{C}$ is complete, then $A \to A/\Gamma^2_\pi(A)$ is a $\mathcal{C}$-isomorphism.*

Proof. Let k be the first integer such that $\Gamma^k_\pi(A) \notin \mathcal{C}$ and $\Gamma^{k+1}_\pi(A) \in \mathcal{C}$. If $A \in \mathcal{C}$ there is nothing to prove. So we can assume that $k \geq 1$. In fact, to show the result, it suffices to show that $k = 1$. Consider the sequence for $k \geq 2$:

$$O \to \Gamma^k_\pi(A) \to \Gamma^{k-1}_\pi(A) \to \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)} \to 0 .$$

From the above sequence of coefficients we get the long exact sequence

$$H_1\left(\pi, \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)}\right) \to H_0(\pi, \Gamma^k_\pi(A)) \to H_0(\pi, \Gamma^{k-1}_\pi(A)) \to H_0\left(\pi, \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)}\right) \to 0$$

But

$$H_0(\pi, \Gamma^{k-1}_\pi(A)) \to H_0\left(\pi, \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)}\right)$$

is an isomorphism. So the sequence reduces to

$$H_1\left(\pi, \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)}\right) \to H_0(\pi, \Gamma^k_\pi(A)) \to 0$$

where $H_0(\pi, \Gamma^k_\pi(A)) \approx \frac{\Gamma^k_\pi(A)}{\Gamma^{k+1}_\pi(A)}$.

Now we will show that $H_1(\pi, C) \in \mathcal{C}$ where $C = \frac{\Gamma^{k-1}_\pi(A)}{\Gamma^k_\pi(A)}$. Certainly π acts trivially on C. Suppose that C is a finitely generated abelian group. Then we have a short exact sequence $0 \to F_1 \to F_2 \to C \to 0$ where F_1, F_2 are finitely generated free abelian groups. By the hypothesis that $\mathcal{C}$ is acyclic follows that $H_*(\pi, F_i) \in \mathcal{C}$. By the long exact sequence

$$\to H_i(\pi, F_1) \to H_i(\pi, F_2) \to H_i(\pi, C) \to H_{i-1}(\pi, F_1)$$

and the fact that $H_0(\pi, F_1) \rightarrow H_0(\pi, F_2)$ is injective follows that $H_*(\pi, C) \in \mathcal{C}$ for $* > 0$.

Now let C be any abelian group and $\mathcal{C}$ complete. Then C is the direct limit of the finitely generated subgroups of C. Since direct limit commutes with homology and $\mathcal{C}$ is closed under direct limit we have that $H_*(\pi, C) \in \mathcal{C}$ in particular $H_1(\pi, C) \in \mathcal{C}$. Therefore

$$H_0(\pi, T_\pi^k(A)) \approx \frac{T_\pi^k(A)}{T_\pi^{k+1}(A)} \in \mathcal{C}.$$

But $T_\pi^{k+1}(A) \in \mathcal{C}$ therefore $T_\pi^k(A) \in \mathcal{C}$ which is a contradiction. So k must be equal to 1 and the result follows. □

Now we will prove a lemma which takes care of the case when the group A is not abelian which corresponds the case $A = \pi_2(Y, X)$.

So we will assume that there is a central extension $0 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$ where $H \subset \pi$.

Lemma 4. *Under the same hypotheses as Lemma 3 we have that $G \rightarrow G/T_\pi^2(G)$ is a $\mathcal{C}$-isomorphism.*

Proof. Let us consider the short exact sequence $0 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$. By the five term long exact sequence, see [S3], we have

$$H_2(G) \rightarrow H_2(H) \rightarrow \frac{A}{[A, G]} \rightarrow G_{ab} \rightarrow H_{ab} \rightarrow 1$$

But $H_2(H) \in \mathcal{C}$, because $\mathcal{C}$ is acyclic, and $[A, G] = 1$ because the extension is central. So $A \rightarrow G_{ab}$ is a $\mathcal{C}$-isomorphism. By Lemma 3, $A \rightarrow \frac{A}{T_\pi^2(A)}$ as well

$$G_{ab} \rightarrow \frac{G_{ab}}{T_\pi^2(G_{ab})} \approx \frac{G}{T_\pi^2(G)}$$

are $\mathcal{C}$-isomorphisms and the result follows.

Proof of Proposition 2. This follows directly from Lemma 3 and Lemma 4. Recall that if a) holds than $\pi_n(Y, X)$ is finitely generated by "The Relative Hurewicz Theorem", p.55 of [G]. □

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